

# EXTENDING REPRESENTATIONS OF BANACH ALGEBRAS TO THEIR BIDUALS

EUSEBIO GARDELLA AND HANNES THIEL

**ABSTRACT.** We show that a representation of a Banach algebra  $A$  on a Banach space  $X$  can be extended to a canonical representation of  $A^{**}$  on  $X$  if and only if certain orbit maps  $A \rightarrow X$  are weakly compact. We apply this to study when the essential space of a representation is complemented. This provides a tool to disregard the difference between degenerate and nondegenerate representations on Banach spaces.

As an application we show that a  $C^*$ -algebra  $A$  has an isometric representation on an  $L^p$ -space, for  $p \in [1, \infty) \setminus \{2\}$ , if and only if  $A$  is commutative.

## 1. INTRODUCTION

A *representation* of a Banach algebra  $A$  on a Banach space  $X$  is a multiplicative, bounded, linear map  $\varphi: A \rightarrow \mathcal{B}(X)$ , where  $\mathcal{B}(X)$  denotes the algebra of bounded linear maps on  $X$ . The *essential space* of  $\pi$ , denoted by  $X_\varphi$ , is the smallest closed subspace of  $X$  containing  $\varphi(a)x$ , for every  $a \in A$  and  $x \in X$ . We want to determine when the essential space of  $\varphi$  is complemented, that is, when there exists an idempotent in  $\mathcal{B}(X)$  whose range is  $X_\varphi$ ; see Problem 3.1.

To study this question, we first consider the related problem of extending a representation of  $A$  to its bidual  $A^{**}$ . It was shown by Arens that the multiplication on  $A$  can be extended in two natural ways (called the left and right Arens product) to a multiplication on  $A^{**}$ .

**Problem 1.1.** Given a representation  $\varphi: A \rightarrow \mathcal{B}(X)$ , find conditions that guarantee that there exists a representation  $\tilde{\varphi}: A^{**} \rightarrow \mathcal{B}(X)$  with  $\tilde{\varphi} \circ \kappa_A = \varphi$ , that is, such that following diagram commutes:

$$\begin{array}{ccc} & A^{**} & \\ \kappa_A \uparrow & \searrow \tilde{\varphi} & \\ A & \xrightarrow{\varphi} & \mathcal{B}(X). \end{array}$$

As stated, this problem is very general and a complete answer seems out of reach. However, in Section 2 we obtain a satisfying answer if we require the extension to be of a specific form. In [GT16a], we introduced a natural multiplication on  $\mathcal{B}(X, X^{**})$  (making it anti-isomorphic to  $\mathcal{B}(X^*)$ ) and a multiplicative operator  $\alpha_X: \mathcal{B}(X)^{**} \rightarrow \mathcal{B}(X, X^{**})$ . We obtain a multiplicative operator  $\alpha_X \circ \varphi^{**}: A^{**} \rightarrow \mathcal{B}(X, X^{**})$ ; see Paragraph 2.1 for details. The main result of the paper is Theorem 2.5, where we show that the image of  $\alpha_X \circ \varphi^{**}$  is contained in  $\mathcal{B}(X)$  (in which case it is the desired canonical extension) if and only if certain orbit maps  $A \rightarrow X$  are weakly compact.

The easiest situation to apply Theorem 2.5 is when *every* operator  $A \rightarrow X$  is weakly compact. In this case, *every* representation of  $A$  on  $X$  has a canonical extension to a representation of  $A^{**}$  on  $X$ ; see Corollary 2.6. There are many cases

---

*Date:* 3 March 2017.

*2010 Mathematics Subject Classification.* Primary: 47L10, Secondary: 43A65, 46E30.

in which every operator  $A \rightarrow X$  is weakly compact: If  $A$  or  $X$  is reflexive; or if  $A$  is a  $C^*$ -algebra and  $X$  does not contain an isomorphic copy of  $c_0$ ; see Paragraph 2.7.

In Section 3, we use these results to study the existence of projections onto the essential space of a representation. We use that  $A^{**}$  has a left unit if and only if  $A$  has a bounded left approximate identity; see Paragraph 3.10. Thus, if the representation  $\varphi: A \rightarrow \mathcal{B}(X)$  has an extension to a representation  $\tilde{\varphi}: A^{**} \rightarrow \mathcal{B}(X)$  then a natural candidate for a projection onto the essential space is given by the image of a left unit in  $A^{**}$  under  $\tilde{\varphi}$ , although the range of this projection could be strictly bigger than the essential space of  $\varphi$ .

We show that this is not the case for the canonical extensions studied in Section 2: If every operator  $A \rightarrow X$  is weakly compact, and if  $A$  contains a bounded left approximate identity, then the essential space of every representation of  $A$  on  $X$  is complemented; see Theorem 3.12. We also obtain bounds on the norm of the projection which allow us to show that the essential space is 1-complemented under the additional assumption that  $\varphi$  is contractive and  $A$  has a contractive left approximate identity; see Corollary 3.13.

In Section 4 we present the main application of our results: A  $C^*$ -algebra can be isometrically represented on some  $L^p$ -space, for  $p \in [1, \infty) \setminus \{2\}$ , if and only if it is commutative; see Theorem 4.4. This result is somewhat surprising at first sight, and it should be compared with the fact that every  $C^*$ -algebra can be represented on a *noncommutative*  $L^p$ -space, for any  $p \in [1, \infty)$ ; see Remarks 4.5.

To prove Theorem 4.4, we first use the results from Section 3 to reduce to the unital case. For unital representations, we then use Lamperti's theorem to show that all hermitian operators on an  $L^p$ -space, for  $p \neq 2$ , commute; see Lemma 4.3.

In Section 5 we study universal completions of Banach algebras with respect to representations on certain classes of Banach spaces. For example, given a locally compact group  $G$ , we consider the Banach algebra  $L^1(G)$  with product given by convolution. Nondegenerate representations of  $L^1(G)$  on a Banach space  $X$  are in natural bijection with isometric representations of  $G$  on  $X$ . Fix  $p \in (1, \infty)$ . The completion of  $L^1(G)$  for nondegenerate representations on  $L^p$ -spaces is called the *universal group  $L^p$ -operator algebra*, denoted  $F^p(G)$ . The Banach algebra  $F^p(G)$  captures the representation theory of  $G$  on  $L^p$ -spaces. For  $p = 2$  we have  $F^p(G) = C^*(G)$ , the universal group  $C^*$ -algebra.

We use our results from Section 3 to show that  $F^p(G)$  is also universal for degenerate representations of  $L^1(G)$  on  $L^p$ -spaces; see Corollary 5.3. This means that the restriction to *nondegenerate* representations of  $L^1(G)$  is unnecessary.

The results of this paper, in particular Theorem 4.4 and Corollary 5.3, have been applied in [GT16c] to give a complete answer to the question when certain algebras of convolution operators on  $L^p(G)$ , for some locally compact group  $G$ , are representable on  $L^q$ -spaces; see also Remark 5.4.

#### ACKNOWLEDGEMENTS

The authors would like to thank Philip G. Spain for valuable electronic correspondence. Part of this research was conducted while the authors were taking part in the Research Program *Classification of operator algebras, complexity, rigidity and dynamics*, held at the Institut Mittag-Leffler, between January and April of 2016. The authors would like to thank the staff and organizers, and Søren Eilers in particular, for the hospitality during their visits. The first named author was partially supported by a Postdoctoral Research Fellowship from the Humboldt Foundation. The authors were partially supported by the Deutsche Forschungsgemeinschaft (SFB 878).

## NOTATION AND TERMINOLOGY

We follow the notation in [GT16a]. An *operator* between Banach spaces means a bounded, linear map. Given a Banach space  $X$ , we let  $\mathcal{B}(X)$  denote the Banach algebra of operators on  $X$ . We let  $\kappa_X: X \rightarrow X^{**}$  denote the natural isometric map from  $X$  to its bidual. Given also a Banach algebra  $A$ , a *representation* of  $A$  on  $X$  is a multiplicative operator  $\varphi: A \rightarrow \mathcal{B}(X)$ . The *essential space* of a representation  $\varphi$  is defined as  $X_\varphi := \overline{\text{span}}\varphi(A)X$ , the closed subspace of  $X$  generated by the ranges of the operators  $\varphi(a)$  for  $a \in A$ . We say that  $\varphi$  is *nondegenerate* if  $X_\varphi = X$ . If  $A$  is unital, then  $\varphi$  is nondegenerate if and only if  $\varphi$  is unital.

## 2. EXTENDING REPRESENTATIONS OF A BANACH ALGEBRA TO ITS BIDUAL

Throughout this section,  $A$  denotes a Banach algebra,  $X$  denotes a Banach space, and  $\varphi: A \rightarrow \mathcal{B}(X)$  denotes a representation.

Using the multiplicative operator  $\alpha_X: \mathcal{B}(X)^{**} \rightarrow \mathcal{B}(X, X^{**})$  constructed in [GT16a], we extend  $\varphi$  to a representation  $\tilde{\varphi}: A^{**} \rightarrow \mathcal{B}(X, X^{**})$ ; see Paragraph 2.1 and Proposition 2.3. We characterize when the image of  $\tilde{\varphi}$  is contained in  $\mathcal{B}(X)$  in terms of weak compactness of orbit maps  $A \rightarrow X$ ; see Theorem 2.5.

If every operator  $A \rightarrow X$  is weakly compact, then every representation of  $A$  on  $X$  can be extended to  $A^{**}$ ; see Corollary 2.6.

**2.1.** Given operators  $a, b: X \rightarrow X^{**}$ , their product  $ab$  is defined as the composition  $\kappa_{X^*}^* \circ a^{**} \circ b$ . This gives  $\mathcal{B}(X, X^{**})$  the structure of a unital Banach algebra, with unit  $\kappa_X$ ; see [GT16a, Paragraph 4.9]. Note that  $\mathcal{B}(X, X^{**})$  is *anti-isomorphic* to  $\mathcal{B}(X^*)$ ; see [GT16a, Proposition 4.10]

We let  $\gamma_X: \mathcal{B}(X) \rightarrow \mathcal{B}(X, X^{**})$  be given by  $\gamma_X(a) := \kappa_X \circ a$  for  $a \in \mathcal{B}(X)$ . Then  $\gamma_X$  is an isometric, multiplicative operator; see [GT16a, Paragraph 4.14].

Let  $\alpha_X: \mathcal{B}(X)^{**} \rightarrow \mathcal{B}(X, X^{**})$  be the contractive map introduced in [GT16a, Definition 4.17]. Instead of recalling the definition of  $\alpha_X$ , we provide a formula to compute  $\alpha_X(S)$  for  $S \in \mathcal{B}(X)^{**}$ ; see Lemma 2.2. (For the purposes of this paper, Lemma 2.2 can be taken as the definition of  $\alpha_X$ .)

The map  $\alpha_X$  is multiplicative (for either Arens product on  $\mathcal{B}(X)^{**}$ ), and we have  $\gamma_X = \alpha_X \circ \kappa_{\mathcal{B}(X)}$ ; see Theorem 4.22 and Corollary 4.24 in [GT16a]. We thus obtain the following commutative diagram:

$$\begin{array}{ccccc} A^{**} & \xrightarrow{\varphi^{**}} & \mathcal{B}(X)^{**} & \xrightarrow{\alpha_X} & \mathcal{B}(X, X^{**}) \\ \kappa_A \uparrow & & \kappa_{\mathcal{B}(X)} \uparrow & \nearrow \gamma_X & \\ A & \xrightarrow{\varphi} & \mathcal{B}(X) & & \end{array}$$

The following result is the case  $X = Y$  of [GT16a, Lemma 4.19].

**Lemma 2.2.** *Let  $X$  be a Banach space. Let  $\alpha_X: \mathcal{B}(X)^{**} \rightarrow \mathcal{B}(X, X^{**})$  be the map from [GT16a, Definition 4.17]. For  $x \in X$ , let  $\text{ev}_x: \mathcal{B}(X) \rightarrow X$  be the evaluation map given by  $\text{ev}_x(a) = ax$ , for  $a \in \mathcal{B}(X)$ . We consider the bitranspose  $\text{ev}_x^{**}: \mathcal{B}(X)^{**} \rightarrow X^{**}$ . Then*

$$\alpha_X(S)x = \text{ev}_x^{**}(S),$$

for all  $S \in \mathcal{B}(X)^{**}$  and  $x \in X$ .

**Proposition 2.3.** *Let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a representation. Then the map  $\tilde{\varphi} := \alpha_X \circ \varphi^{**}: A^{**} \rightarrow \mathcal{B}(X, X^{**})$  is multiplicative for either Arens product on  $\mathcal{B}(X)^{**}$ . Moreover, we have  $\|\tilde{\varphi}\| = \|\varphi\|$ .*

*Proof.* In general, the bitranspose of a multiplicative map between Banach algebras is again multiplicative if both bidual Banach algebras are equipped with the same Arens product. Thus,  $\varphi^{**}$  is multiplicative if  $A^{**}$  and  $\mathcal{B}(X)^{**}$  are both equipped with the same Arens product. By [GT16a, Corollary 4.24],  $\alpha_X$  is multiplicative for either Arens product on  $\mathcal{B}(X)^{**}$ , which implies the first statement.

By [GT16a, Remark 4.18], the map  $\alpha_X$  is contractive. It follows that  $\|\tilde{\varphi}\| \leq \|\varphi\|$ . The converse estimate follows using that  $\gamma_X \circ \varphi = \tilde{\varphi} \circ \kappa_A$  (see the diagram in Paragraph 2.1) and that  $\gamma_X$  is isometric.  $\square$

Recall that an operator  $f: X \rightarrow Y$  is said to be *weakly compact* if the image of the unit ball of  $X$  is contained in a weakly compact subset of  $Y$ . By Gantmacher's theorem,  $f$  is weakly compact if and only if its transpose  $f^*$  is. Moreover,  $f$  is weakly compact if and only if the bitranspose  $f^{**}: X^{**} \rightarrow Y^{**}$  takes image in  $Y$ , that is,  $F^{**}(X^{**}) \subseteq Y$ ; see [Con90, Theorem VI.5.5, p.185].

**Lemma 2.4.** *Let  $X$  and  $Y$  be Banach spaces, let  $Y_0 \subseteq Y$  be a closed subspace, and let  $f: X \rightarrow Y$  be an operator whose image is contained in  $Y_0$ . Then  $f$  is weakly compact if and only if the image of  $f^{**}: X^{**} \rightarrow Y^{**}$  is contained in  $Y_0$ .*

*Proof.* If the image of  $f^{**}$  is contained in  $Y_0$ , then it is also contained in  $Y$ , and consequently  $f$  is weakly compact. To show the backward implication, assume that  $f$  is weakly compact. Let  $\iota: Y_0 \rightarrow Y$  be the inclusion map. Using that the image of  $f$  is contained in  $Y_0$ , we let  $f_0: X \rightarrow Y_0$  be the unique map such that  $f = \iota \circ f_0$ .

The bitranspose  $\iota^{**}: Y_0^{**} \rightarrow Y^{**}$  is isometric. We use it to identify  $Y_0^{**}$  with a subspace of  $Y^{**}$ . Similarly, we consider  $Y$  as a subspace of  $Y^{**}$  (via  $\kappa_Y$ ). There is a commutative diagram:

$$\begin{array}{ccccc} X^{**} & \xrightarrow{f_0^{**}} & Y_0^{**} & \xrightarrow{\iota^{**}} & Y^{**} \\ \kappa_X \uparrow & & \kappa_{Y_0} \uparrow & & \kappa_Y \uparrow \\ X & \xrightarrow{f_0} & Y_0 & \xrightarrow{\iota} & Y \end{array}$$

Claim: We have  $Y_0 = Y_0^{**} \cap Y$  in  $Y^{**}$ .

The inclusion ' $\subseteq$ ' is clear. To show the converse inclusion, let  $y \in Y_0^{**} \cap Y$ . Choose a net  $(y_j)_j$  in  $Y_0$  that converges weak\* to  $y$  in  $Y_0^{**}$ . Since the inclusion  $\iota^{**}$  is weak\* continuous, we deduce that  $(y_j)_j$  converges weak\* to  $y$  in  $Y^{**}$ . Since the net  $(y_j)_j$  and  $y$  belong to  $Y$ , it follows that  $(y_j)_j$  converges weakly to  $y$ . It follows from the Hahn-Banach theorem that (norm-)closed subspaces are weakly closed, which implies that  $y$  belongs to  $Y_0$ , which proves the claim.

Since  $f$  is weakly compact, the image of  $f^{**}$  is contained in  $Y$ . Using that  $f = \iota \circ f_0$ , and that  $\iota^{**}$  is isometric, it follows that the image of  $f^{**}$  is contained in the image of  $f_0^{**}$ , which in turn is contained in  $Y_0^{**}$ . Using the claim, it follows that the image of  $f^{**}$  is contained in  $Y_0$ , as desired.  $\square$

**Theorem 2.5.** *Let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a representation. Set  $\tilde{\varphi} := \alpha_X \circ \varphi^{**}: A^{**} \rightarrow \mathcal{B}(X, X^{**})$ , as in Proposition 2.3. Then the image of  $\tilde{\varphi}$  is contained in  $\mathcal{B}(X)$  if and only if for every  $x \in X$  the orbit map  $\text{ev}_x \circ \varphi: A \rightarrow X$  is weakly compact. Moreover, if this is the case, then the essential space of  $\varphi$  and  $\tilde{\varphi}$  agree.*

*Proof.* Using Lemma 2.2 at the first step, we have

$$\tilde{\varphi}(S)x = \text{ev}_x^{**}(\varphi^{**}(S)) = (\text{ev}_x \circ \varphi)^{**}(S),$$

for all  $S \in A^{**}$  and  $x \in X$ . Recall that an operator  $f: E \rightarrow F$  is weakly compact if and only if the image of  $f^{**}$  is contained in  $F \subseteq F^{**}$ . Thus, given  $x \in X$ , the

orbit map  $\text{ev}_x \circ \varphi$  is weakly compact if and only if  $\tilde{\varphi}(S)x$  belongs to  $\mathcal{B}(X)$  for every  $S \in A^{**}$ . This implies the first statement.

To show the second statement, assume that every orbit map  $\text{ev}_x \circ \varphi$  is weakly compact. Let  $X_\varphi$  and  $X_{\tilde{\varphi}}$  be the essential spaces of  $\varphi$  and  $\tilde{\varphi}$ , respectively. We clearly have  $X_\varphi \subseteq X_{\tilde{\varphi}}$ . To show the converse containment, let  $x \in X$ . Since  $\text{ev}_x \circ \varphi$  is weakly compact with image contained in  $X_\varphi$ , it follows from Lemma 2.4 that the image of  $(\text{ev}_x \circ \varphi)^{**}$  is contained in  $X_\varphi$  as well. Using Lemma 2.2 at the first step, we deduce that

$$\tilde{\varphi}(S)x = (\text{ev}_x \circ \varphi)^{**}(S) \in X_\varphi,$$

for every  $S \in A^{**}$  and  $x \in X$ . This implies that the essential space of  $\tilde{\varphi}$  is contained in  $X_\varphi$ , as desired.  $\square$

The following consequence is our desired (partial) answer to Problem 1.1. It gives a sufficient condition to be able to extend, in a canonical way, every representation  $A \rightarrow \mathcal{B}(X)$  to a representation  $A^{**} \rightarrow \mathcal{B}(X)$ .

**Corollary 2.6.** *Let  $A$  be a Banach algebra, let  $X$  be a Banach space such that every operator  $A \rightarrow X$  is weakly compact, and let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a representation. Set  $\tilde{\varphi} := \alpha_X \circ \varphi^{**}: A^{**} \rightarrow \mathcal{B}(X, X^{**})$ , as in Proposition 2.3. Then  $\tilde{\varphi}$  takes image in  $\mathcal{B}(X)$ , and therefore  $\tilde{\varphi}$  is an extension of  $\varphi$  to  $A^{**}$ . Moreover, the essential spaces of  $\varphi$  and  $\tilde{\varphi}$  agree.*

**2.7.** Let  $X$  and  $Y$  be Banach spaces. Let us recall sufficient conditions on  $X$  and  $Y$  to ensure that every operator  $X \rightarrow Y$  is weakly compact.

First, it is well-known that a Banach space  $E$  is reflexive if and only if  $\text{id}_E$  is weakly compact. Second, the composition  $f \circ g$  of two operators  $f$  and  $g$  is weakly compact as soon as  $f$  or  $g$  is weakly compact. Thus, if  $X$  or  $Y$  is reflexive, then every operator  $X \rightarrow Y$  is weakly compact.

However, there are other situations where neither  $X$  nor  $Y$  is reflexive, yet every operator  $X \rightarrow Y$  is weakly compact. This is for instance the case when  $X$  has Pełczyński's property (V) and  $Y$  contains no isomorphic copy of  $c_0$ . Let us recall some details.

A series  $\sum_n x_n$  in  $X$  is said to be (weakly) *unconditionally convergent*, abbreviated (w.)u.c., if for every permutation  $(k_n)_n$  of indices, the series  $\sum_n x_{k_n}$  converges (weakly). Further, an operator  $T: X \rightarrow Y$  is said to be *unconditionally convergent* if it sends every w.u.c. series in  $X$  to a u.c. series in  $Y$ . By a result of Bessaga and Pełczyński, [BPc58, Theorem 5], a space  $Y$  contains no isomorphic copy of  $c_0$  if and only if for every Banach space  $X$ , every operator  $X \rightarrow Y$  is unconditionally convergent.

A Banach space  $X$  is said to have *Pełczyński's property (V)* if for every Banach space  $Y$ , every unconditionally convergent operator  $X \rightarrow Y$  is weakly compact. Therefore, if  $X$  has Pełczyński's property (V) and  $Y$  contains no isomorphic copy of  $c_0$ , then every operator  $X \rightarrow Y$  is weakly compact.

By a deep result of Pfitzner, [Pfi94, Corollary 6], every  $C^*$ -algebra  $A$  has property (V). It follows that if  $Y$  contains no isomorphic copy of  $c_0$  then every operator  $A \rightarrow Y$  is weakly compact. This had previously been obtained (without showing the stronger result that  $C^*$ -algebras have property (V)) in [ADG72, Theorem 4.2], see also [Spa76].

**Corollary 2.8.** *Let  $A$  be a  $C^*$ -algebra and let  $X$  be a Banach space that does not contain an isomorphic copy of  $c_0$ . Then every representation  $\varphi: A \rightarrow \mathcal{B}(X)$  can be extended to  $A^{**}$ .*

**Remark 2.9.** For unital representations of unital  $C^*$ -algebras, Corollary 2.8 has implicitly been obtained by Spain, [Spa15, Section 9]. For our applications in

Section 3 we have to consider nonunital Banach algebras and nonunital representations.

### 3. PROJECTIONS ONTO THE ESSENTIAL SPACE OF A REPRESENTATION

Given a Banach space  $X$  and a closed subspace  $X_0 \subseteq X$ , recall that a *projection* from  $X$  onto  $X_0$  is an idempotent operator on  $X$  with image  $X_0$ .

In this section, we study the following:

**Problem 3.1.** Let  $A$  be a Banach algebra, and let  $X$  be a Banach space. Under which assumptions on  $A$  and  $X$  does it follow that for every representation  $\varphi: A \rightarrow \mathcal{B}(X)$  there exists a projection  $p$  onto the essential space of  $\varphi$ ? Moreover, if such a projection  $p$  exists, can we estimate  $\|p\|$ ?

**Remark 3.2.** Recall that a closed subspace  $X_0$  of a Banach space  $X$  is called *complemented* if there exists a projection  $p$  of  $X$  onto  $X_0$ . Moreover, for  $\lambda \in \mathbb{R}_+$ , we say  $X_0$  is  $\lambda$ -*complemented* in  $X$  if it is the range of a projection  $p$  with  $\|p\| \leq \lambda$ . The infimum of such  $\lambda$  is called the *relative projection constant* of  $X_0$  in  $X$ , denoted  $\lambda(X_0, X)$ .

Thus, Problem 3.1 is asking to determine conditions under which the essential space  $X_0$  of a representation is automatically complemented, and to give estimates on the relative projection constant.

Of course, the answer to Problem 3.1 depends crucially on  $A$  and  $X$ . The better behaved  $A$  is, the fewer assumptions we need to put on  $X$ , and conversely. The extreme points for this tradeoff are marked by the case that  $X$  is isomorphic to a Hilbert space, and on the other end that  $A$  has a left unit; see Remarks 3.3.

It becomes a natural question to ask if every Banach algebra without (left) unit has a representation whose essential space is not complemented; see Question 3.5. This question is already nontrivial for nonunital  $C^*$ -algebras; see Example 3.7 and Remarks 3.9.

The main result of this section is Theorem 3.12, which provides a positive solution to Problem 3.1 for the case that  $A$  has a bounded left approximate identity and every operator  $A \rightarrow X$  is weakly compact. We observe that an isometric, nondegenerate representation of a Banach algebra  $A$  with a contractive, left approximate identity can be extended to an isometric, unital representation of the (left) multiplier algebra  $M_l(A)$ ; see Theorem 3.17. Combining these results, we show that if  $A$  is an  $L^p$ -operator algebra with a contractive, left approximate identity, then the left multiplier algebra  $M_l(A)$  is an  $L^p$ -operator algebra as well; see Theorem 3.19.

**Remarks 3.3.** (1) If  $X$  is a Hilbert space, then every closed subspace of  $X$  is 1-complemented. More generally, if  $X$  is isomorphic to a Hilbert space  $H$  via an isomorphism  $\alpha$ , then every closed subspace of  $X$  is  $\|\alpha\| \cdot \|\alpha^{-1}\|$ -complemented. Hence, for such  $X$ , for every Banach algebra  $A$  and for every representation  $\varphi$  of  $A$  on  $X$ , the essential space of  $\varphi$  is complemented. Thus, we consider Problem 3.1 solved in the case that  $X$  is isomorphic to a Hilbert space.

If  $X$  is not isomorphic to a Hilbert space, then there always exists a representation  $\varphi$  on  $X$  such the essential space of  $\varphi$  is not complemented; see Example 3.4. Thus, the only Banach spaces such that the solution to Problem 3.1 is positive regardless of the Banach algebra are the Banach spaces isomorphic to Hilbert spaces.

(2) If  $A$  contains a left unit  $e$  (in particular, if  $A$  is unital), then for every Banach space  $X$  and every representation  $\varphi: A \rightarrow \mathcal{B}(X)$  the element  $p := \varphi(e)$  is a projection onto the essential space of  $\varphi$ . Moreover, we have  $\|p\| \leq \|\varphi\| \cdot \|e\|$ . Thus, we consider Problem 3.1 solved in the case that  $A$  contains a left unit.

Thus, for Banach algebras with left unit, the answer to Problem 3.1 is positive regardless of the Banach space. It is conceivable that this characterizes Banach algebras with left unit; see Question 3.5.

**Example 3.4.** By the Lindenstrauss-Tzafriri theorem, [LT71], a Banach space is isomorphic to a Hilbert space if and only if all of its closed subspaces are complemented. Thus, given a Banach space  $X$  that is not isomorphic to a Hilbert space, we can choose a closed subspace  $X_0$  that is not complemented in  $X$ .

We claim that  $X_0$  is the essential space of an isometric representation of a Banach algebra on  $X$ . Indeed, consider  $\mathcal{B}(X, X_0)$ , which we identify with the space of operators  $a \in \mathcal{B}(X)$  satisfying  $a(X) \subseteq X_0$ . Then  $\mathcal{B}(X, X_0)$  is a closed subalgebra of  $\mathcal{B}(X)$ . We consider the inclusion map as an isometric representation  $\varphi: \mathcal{B}(X, X_0) \rightarrow \mathcal{B}(X)$ . Using rank-one operators, we get that the essential space of  $\varphi$  is exactly  $X_0$ .

**Question 3.5.** Let  $A$  be a Banach algebra without left unit. Does there exist a representation  $\varphi: A \rightarrow \mathcal{B}(X)$  on some Banach space  $X$  such that the essential space of  $\varphi$  is not complemented?

**Example 3.6.** Let  $A := c_0$  be the  $C^*$ -algebra of sequences that converge to zero. Let  $X := \ell_\infty$  be the Banach space of bounded sequences. Then  $A$  acts on  $X$  by pointwise multiplication. This defines an isometric representation

$$\varphi: c_0 \rightarrow \mathcal{B}(\ell_\infty).$$

The essential space of  $\varphi$  is exactly the subspace  $c_0$  in  $\ell_\infty$ . However, Phillips' theorem states that  $c_0$  is not complemented in  $\ell_\infty$ . (See [Whi66] for a simple proof of Phillips' theorem.) This provides a positive answer to Question 3.5 for  $A = c_0$ .

**Example 3.7.** We can generalize Example 3.6 to nonunital  $C^*$ -algebras. Note that  $\ell_\infty$  can be identified with the multiplier algebra of  $c_0$ . In general, if  $A$  is a (nonunital)  $C^*$ -algebra, then the multiplier algebra  $M(A)$  is a unital  $C^*$ -algebra with a certain universal property and such that there is a canonical, isometric  $*$ -homomorphism  $\iota: A \rightarrow M(A)$  that identifies  $A$  with a closed, two-sided ideal in  $M(A)$ . We can therefore consider the representation  $\varphi_A: A \rightarrow \mathcal{B}(M(A))$ , where  $a \in A$  acts on  $b \in M(A)$  simply by  $\varphi_A(a)b := \iota(a)b$  (multiplication of  $\iota(a)$  and  $b$  in  $M(A)$ ). As in the case of  $A = c_0$ , the essential space of this representation is exactly the subspace  $A$  in  $M(A)$ . Thus, we obtain a positive answer to Question 3.5 for every  $C^*$ -algebra  $A$  that is not complemented in its multiplier algebra. (This includes all  $\sigma$ -unital  $C^*$ -algebras; see Remarks 3.9(1).) We therefore ask:

**Question 3.8.** Let  $A$  be a nonunital  $C^*$ -algebra. Is  $A$  an uncomplemented subspace of its multiplier algebra  $M(A)$ ?

**Remarks 3.9.** (1) Let  $A$  be a nonunital  $C^*$ -algebra. Taylor showed in [Tay72, Corollary 3.7] that  $A$  is uncomplemented in  $M(A)$  if  $A$  has a well-behaved approximate identity. (See [Tay72] for the definition.) Recall that  $A$  is  $\sigma$ -unital if it contains a strictly positive element, which is equivalent to  $A$  having a countable approximate identity. Moreover, every separable  $C^*$ -algebra is  $\sigma$ -unital; see [Bla06, Section II.4.2, p.81ff] for details.

By [Tay72, Proposition 3.1], every  $\sigma$ -unital  $C^*$ -algebra has a well-behaved approximate identity. Thus, we have a positive answer to Question 3.8, and consequently to Question 3.5, for every  $\sigma$ -unital  $C^*$ -algebra, in particular every separable  $C^*$ -algebra.

(2) Let  $A$  be a nonunital  $C^*$ -algebra. We let  $\tilde{A}$  denote the minimal unitization of  $A$ . As Banach spaces, we have  $\tilde{A} \cong A \oplus \mathbb{C}$ , which shows that  $A$  is always complemented in  $\tilde{A}$ . There exist nonunital (even commutative)  $A$  such that  $\tilde{A} = M(A)$ . For such  $A$ , Question 3.8 has a negative answer. (But the answer to Question 3.5

remains unclear.) For examples and a further discussion of  $C^*$ -algebra whose multiplier algebra agrees with the minimal unitization, we refer to [GK16].

**3.10.** Let  $A$  be a Banach algebra. A *left (right) approximate identity* in  $A$  is a net  $(e_j)_j$  in  $A$  such that  $\lim_j \|e_j a - a\| = 0$  (resp.  $\lim_j \|a e_j - a\| = 0$ ) for each  $a \in A$ . An approximate identity is a net that is both a left and right approximate identity.

Let  $\lambda \in \mathbb{R}_+$ . A (left, right) approximate identity is *bounded*, with bound  $\lambda$ , if  $\|e_j\| \leq \lambda$  for each  $j$ . For  $\lambda = 1$  we also speak of a *contractive* (left, right) approximate identity.

Given a bounded left approximate identity  $(e_j)_j$  in  $A$ , let  $e$  be any weak\* cluster point of the net  $(e_j)_j$  in  $A^{**}$ . Then  $ea = a$  for every  $a \in A$ . Since the second Arens product is weak\* continuous in the second variable, we obtain that  $e \diamond a = a$  for all  $a \in A^{**}$ . Thus,  $e$  is a left unit for the second Arens products. The converse also holds, that is,  $A$  has a bounded left approximate identity (with bound  $\lambda$ ) if and only if  $(A^{**}, \diamond)$  has a left unit (with norm  $\leq \lambda$ ).

Analogously,  $A$  has a bounded right approximate identity if and only if  $(A^{**}, \square)$  has a right unit. Combining these, it follows easily that  $A$  has a bounded approximate identity if and only if there is  $e \in A^{**}$  such that  $e \diamond a = a = a \square e$  for all  $a \in A^{**}$ . (Such  $e$  is called a *mixed unit* for  $A^{**}$ .) We refer to [Pal94, Proposition 5.1.8, p.527] for details. In particular, if  $A$  is Arens regular and has a contractive approximate identity, then  $A^{**}$  is unital with  $\|1_{A^{**}}\| = 1$ .

**Lemma 3.11.** *Let  $A$  be a Banach algebra with a bounded left approximate identity with bound  $\lambda$ , let  $X$  be a Banach space, and let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a representation. Set  $\tilde{\varphi} := \alpha_X \circ \varphi^{**}: A^{**} \rightarrow \mathcal{B}(X, X^{**})$ , as in Proposition 2.3.*

*Let  $e \in A^{**}$  be a left unit for  $(A^{**}, \diamond)$  with  $\|e\| \leq \lambda$ . Set  $p := \tilde{\varphi}(e)$ . Then  $p$  is an idempotent in  $\mathcal{B}(X, X^{**})$  with  $\|p\| \leq \lambda \|\varphi\|$  and such that  $p \tilde{\varphi}(a) = \tilde{\varphi}(a)$  for all  $a \in A^{**}$ .*

*Proof.* By Proposition 2.3, we have  $\|\tilde{\varphi}\| = \|\varphi\|$ , which implies that

$$\|p\| = \|\tilde{\varphi}(e)\| \leq \|\tilde{\varphi}\| \|e\| \leq \|\varphi\| \lambda.$$

By Proposition 2.3,  $\tilde{\varphi}$  is multiplicative for either Arens product on  $A^{**}$ . Thus

$$p^2 = \tilde{\varphi}(e \diamond e) = \tilde{\varphi}(e) = p.$$

Moreover, for every  $a \in A^{**}$ , we have

$$p \tilde{\varphi}(a) = \tilde{\varphi}(e) \tilde{\varphi}(a) = \tilde{\varphi}(e \diamond a) = \tilde{\varphi}(a),$$

as desired.  $\square$

The following consequence is a (partial) solution to Problem 3.1.

**Theorem 3.12.** *Let  $A$  be a Banach algebra with a bounded left approximate identity with bound  $\lambda$ , let  $X$  be a Banach space such that every operator  $A \rightarrow X$  is weakly compact, and let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a representation. Then the essential space of  $\varphi$  is a  $\lambda \|\varphi\|$ -complemented subspace of  $X$ .*

*Proof.* Set  $\tilde{\varphi} := \alpha_X \circ \varphi^{**}: A^{**} \rightarrow \mathcal{B}(X, X^{**})$ , as in Proposition 2.3. It follows from Corollary 2.6 that  $\tilde{\varphi}$  takes image in  $\mathcal{B}(X)$  and its essential space agrees with that of  $\varphi$ . Let  $e \in A^{**}$  be a left unit for  $(A^{**}, \diamond)$  with  $\|e\| \leq \lambda$ , and set  $p := \tilde{\varphi}(e)$ . Then  $p$  is an idempotent in  $\mathcal{B}(X)$  and we have  $\|p\| \leq \lambda \|\varphi\|$  by Lemma 3.11. The range of  $p$  is contained in the essential space of  $\tilde{\varphi}$ . Conversely, it follows from Lemma 3.11 that  $p$  acts as the identity on the essential space of  $\tilde{\varphi}$ . Thus,  $p$  is the desired projection onto the essential space of  $\varphi$ .  $\square$



**Corollary 3.13.** *Let  $A$  be a Banach algebra with a contractive left approximate identity, let  $X$  be a reflexive Banach space, and let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a contractive representation. Then the essential space of  $\varphi$  is 1-complemented in  $X$ .*

*Proof.* Since  $X$  is reflexive, every operator  $A \rightarrow X$  is weakly compact. Thus, we may apply Theorem 3.12.  $\square$

**Remark 3.14.** The class of spaces that do not contain an isomorphic copy of  $c_0$  has been studied in [BPc58]. It includes all reflexive space, and all  $L^p$ -spaces for  $p \in [1, \infty)$ ; see Lemma 4.1.

**Proposition 3.15.** *Let  $A$  be a  $C^*$ -algebra, let  $X$  be a Banach space that does not contain an isomorphic copy of  $c_0$ , and let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a contractive representation. Then the essential space of  $\varphi$  is 1-complemented in  $X$ .*

*Proof.* The assumptions imply that every operator  $A \rightarrow X$  is weakly compact; see Paragraph 2.7. Moreover, every  $C^*$ -algebra has a contractive approximate identity. Thus, we may apply Theorem 3.12 to deduce the statement.  $\square$

**3.16.** Let  $A$  be a Banach algebra. Let us recall some details about the (left) multiplier algebra of  $A$ . For more details and further generalizations, we refer to the original paper by Johnson, [Joh64], and recent work by Daws, [Daw10].

A *left multiplier* on  $A$  is a operator  $L: A \rightarrow A$  such that  $L(ab) = L(a)b$  for all  $a, b \in A$ . The *left multiplier algebra* of  $A$ , denoted by  $M_l(A)$ , is the collection of all left multipliers on  $A$ . We can think of  $M_l(A)$  as a subset of  $\mathcal{B}(A)$ . It is easy to see that  $M_l(A)$  is a unital, closed subalgebra of  $\mathcal{B}(A)$ , which gives  $M_l(A)$  the structure of a unital Banach algebra.

Every  $a \in A$  defines a left multiplier  $L_a \in M_l(A)$  by  $L_a(x) := ax$ , for  $x \in A$ . This defines a multiplicative operator  $L: A \rightarrow M_l(A)$ . If  $A$  has a contractive approximate identity, then  $L$  is an isometry that embeds  $A$  as a closed, left ideal in  $M_l(A)$ .

A *multiplier* on  $A$  is a pair of operators  $R, L: A \rightarrow A$  such that  $aL(b) = R(a)b$  for all  $a, b \in A$ . The *multiplier algebra* of  $A$ , denoted by  $M(A)$ , is the collection of all multipliers  $(L, R)$  on  $A$ . Given  $(L_1, R_1), (L_2, R_2) \in M(A)$ , we set

$$(L_1, R_1) + (L_2, R_2) := (L_1 + L_2, R_1 + R_2), \quad \text{and} \quad (L_1, R_1)(L_2, R_2) := (L_2 L_1, R_1 R_2).$$

Further, we define the norm as  $\|(L, R)\| := \max\{\|L\|, \|R\|\}$ . This gives  $M(A)$  the structure of a unital Banach algebra, with unit  $(\text{id}_A, \text{id}_A)$ .

Every  $a \in A$  defines a multiplier  $(L_a, R_a) \in M(A)$  by  $L_a(x) := ax$  and  $R_a(x) := xa$ , for  $x \in A$ . This defines a multiplicative operator  $\iota: A \rightarrow M(A)$ . If  $A$  has a contractive approximate identity, then  $\iota$  is an isometry that embeds  $A$  as a closed, two-sided ideal in  $M(A)$ . Moreover, for every  $(L, R) \in M(A)$ , the operator  $L$  is a left multiplier. This induces a multiplicative isometry  $M(A) \rightarrow M_l(A)$ ,  $(L, R) \mapsto L$ , which identifies the multiplier algebra with a unital, closed subalgebra of the left multiplier algebra.

If  $A$  is a  $C^*$ -algebra, then so is  $M(A)$  with involution given by  $(L, R)^* := (R^*, L^*)$  for  $L^*: A \rightarrow A$  defined as  $L^*(a) := L(a^*)^*$ , for  $a \in A$ , and similarly for  $R^*$ ; see [Bla06, II.7.3.4, p.145].

It is well-known that a nondegenerate representation of a Banach algebra  $A$  with bounded approximate identity can be extended to a representation of the (left) multiplier algebra  $M_l(A)$ ; see for example [Daw10, Theorem 3.2]. For our purposes, we need to show that a nondegenerate *isometric* representation of  $A$  can be extended to an unital, *isometric* representation of  $M_l(A)$ . This result is certainly also well-known, but we could not locate a reference. We therefore include an argument.

**Theorem 3.17.** *Let  $X$  be a Banach space, let  $A$  be a Banach algebra with a contractive, left approximate identity, and let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a contractive, nondegenerate representation. Then there is a unique contractive, unital representation  $\tilde{\varphi}: M_l(A) \rightarrow \mathcal{B}(X)$  such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathcal{B}(X) \\ L \downarrow & \nearrow \tilde{\varphi} & \\ M_l(A) & & \end{array}$$

Moreover, for every  $a \in A$ ,  $x \in X$  and  $T \in M_l(A)$ , we have

$$(1) \quad \tilde{\varphi}(T)(\varphi(a)x) = \varphi(T(a))x.$$

Furthermore, if  $\varphi$  is isometric, then so is  $\tilde{\varphi}$ .

*Proof. Step 1.* In order to show that  $\tilde{\varphi}$  is well-defined, let  $T \in M_l(A)$ .

By the Cohen-Hewitt factorization theorem, we have  $X = \varphi(A)X$ . Thus, (1) defines a map  $\tilde{\varphi}(T): X \rightarrow X$ . To show that  $\tilde{\varphi}(T)$  is well-defined let  $a, b \in A$  and  $x, y \in X$  satisfy  $\varphi(a)x = \varphi(b)y$ . Let  $(e_j)_j$  be a contractive, left approximate identity in  $A$ . Using that  $T$  is continuous at the first and last step, and that  $T$  is a left multiplier at the second step, we obtain that

$$\varphi(T(a))x = \lim_j \varphi(T(e_j a))x = \lim_j \varphi(T(e_j) a)x = \lim_j \varphi(T(e_j))(\varphi(a)x).$$

Analogously, we have  $\varphi(T(b))y = \lim_j \varphi(T(e_j))(\varphi(b)y)$ , whence the assumption implies that  $\varphi(T(a))x = \varphi(T(b))y$ , which verifies that  $\tilde{\varphi}$  is well-defined.

**Step 2.** We show that  $\tilde{\varphi}$  is a unital representation satisfying  $\tilde{\varphi} \circ L = \varphi$ .

Given  $S, T \in M_l(A)$ ,  $a \in A$  and  $x \in X$ , we have

$$\begin{aligned} \tilde{\varphi}(ST)(\varphi(a)x) &= \varphi(ST(\varphi(a)))x = \varphi(S(T(\varphi(a))))x \\ &= \tilde{\varphi}(S)(\varphi(T(\varphi(a))))x = \tilde{\varphi}(S)\tilde{\varphi}(T)(\varphi(a)x), \end{aligned}$$

and thus  $\tilde{\varphi}(ST) = \tilde{\varphi}(S)\tilde{\varphi}(T)$ , which shows that  $\tilde{\varphi}$  is multiplicative. Similarly one shows that  $\tilde{\varphi}(\text{id}_A) = \text{id}_X$ . It follows that  $\tilde{\varphi}$  is a unital representation.

Further, given  $a, b \in A$  and  $x \in X$ , we have

$$\tilde{\varphi}(L_a)(\varphi(b)x) = \varphi(L_a(b))x = \varphi(ab)x = \varphi(a)(\varphi(b)x),$$

which implies  $\tilde{\varphi}(L_a) = \varphi(a)$ . Thus,  $\tilde{\varphi} \circ L = \varphi$ , as desired.

**Step 3.** In order to show that  $\tilde{\varphi}$  is contractive, let  $T \in M_l(A)$ .

The Cohen-Hewitt factorization theorem shows that

$$\{x \in X : \|x\| < 1\} = \{\varphi(a)x : a \in A, x \in X, \|a\| \leq 1, \|x\| < 1\}.$$

Using this at the second step, we obtain that

$$\begin{aligned} \|\tilde{\varphi}(T)\| &= \sup \{\|\tilde{\varphi}(T)(x)\| : x \in X, \|x\| < 1\} \\ &= \sup \{\|\tilde{\varphi}(T)(\varphi(a)x)\| : a \in A, x \in X, \|a\| \leq 1, \|x\| < 1\} \\ &= \sup \{\|\varphi(T(a))(x)\| : a \in A, x \in X, \|a\| \leq 1, \|x\| < 1\} \leq \|T\|, \end{aligned}$$

as desired.

**Step 4.** Assume that  $\varphi$  is isometric. In order to show that  $\tilde{\varphi}$  is isometric, let  $T \in M_l(A)$ . Using that  $\varphi$  is isometric at the second step, we compute

$$\begin{aligned} \|T\| &= \sup \{\|T(a)(x)\| : a \in A, x \in X, \|a\| \leq 1, \|x\| \leq 1\} \\ &= \sup \{\|\varphi(T(a))(x)\| : a \in A, x \in X, \|a\| \leq 1, \|x\| \leq 1\} \\ &= \sup \{\|\tilde{\varphi}(T)(\varphi(a)x)\| : a \in A, x \in X, \|a\| \leq 1, \|x\| \leq 1\} \leq \|\tilde{\varphi}(T)\|, \end{aligned}$$

as desired.  $\square$

**Remark 3.18.** The analog of Theorem 3.17 holds with the multiplier algebra in place of the left multiplier algebra. Indeed, the assumptions on  $A$  imply that  $M(A)$  can be identified with a closed subalgebra of  $M_l(A)$ ; see Paragraph 3.16.

Let  $p \in [1, \infty)$ . Recall that a Banach space is called an  $L^p$ -space if it is isometrically isomorphic to  $L^p(\mu)$  for some measure space  $\mu$ . Further, a Banach space is called an  $SL^p$ -space ( $QL^p$ -space,  $QSL^p$ -space) if it is isometrically isomorphic to a subspace (a quotient, a quotient of a subspace) of an  $L^p$ -space.

In Theorem 3.19, we consider classes of reflexive Banach spaces that are closed under passing to 1-complemented subspaces. By [Tza69, Theorem 6], every 1-complemented subspace of an  $L^p$ -space is again an  $L^p$ -space. It follows that for each  $p$ , the class of  $L^p$ -spaces (of  $QL^p$ -spaces, of  $SL^p$ -spaces, of  $QSL^p$ -spaces) is closed under passing to 1-complemented subspaces. An overview of other classes of Banach spaces with this property can be found in [NR11].

Let  $\mathcal{E}$  be a class of Banach spaces. A Banach algebra  $A$  is called an  $\mathcal{E}$ -operator algebra if there exists an isometric representation of  $A$  on some Banach space in  $\mathcal{E}$ . For instance, an  $L^p$ -operator algebra is a Banach algebra that admits an isometric representation on some  $L^p$ -space. Such algebras have for instance been studied in [Phi12], [Phi13], [GT15a], [GT15b], [GT16b].

**Theorem 3.19.** *Let  $\mathcal{E}$  be a class of reflexive Banach spaces that is closed under passing to 1-complemented subspaces (for instance, the class of  $L^p$ -spaces, for some fixed  $p \in (1, \infty)$ ), and let  $A$  be an  $\mathcal{E}$ -operator algebra with a contractive left approximate identity. Then there is a nondegenerate isometric representation of  $A$  on a space in  $\mathcal{E}$ . It follows that the left multiplier algebra of  $A$  has a unital, isometric representation on a space in  $\mathcal{E}$ . In particular,  $M_l(A)$  is a  $\mathcal{E}$ -operator algebra.*

*If, moreover,  $A$  has a contractive approximate identity, then the same statement holds with  $M(A)$  in place of  $M_l(A)$ . In particular,  $M(A)$  is then a  $\mathcal{E}$ -operator algebra.*

*Proof.* Choose  $X$  in  $\mathcal{E}$  and an isometric representation  $\varphi: A \rightarrow \mathcal{B}(X)$ . Let  $X_\varphi$  denote the essential space of  $\varphi$ . By Corollary 3.13,  $X_\varphi$  is 1-complemented in  $X$ . By assumption,  $X_\varphi$  is again in  $\mathcal{E}$ . The restriction of  $\varphi$  to  $X_\varphi$  is the desired nondegenerate representation. Then, we may apply Theorem 3.17 to deduce that the nondegenerate isometric representation of  $A$  on  $X_\varphi$  can be extended to a unital, isometric representation of  $M_l(A)$  on  $X_\varphi$ .

If  $A$  has a contractive approximate identity, then  $M(A)$  can be identified with a unital, closed subalgebra of  $M_l(A)$ ; see Paragraph 3.16. This implies the analog statements for  $M(A)$ .  $\square$

**Remark 3.20.** If  $A$  is a  $C^*$ -algebra, then Theorem 3.19 also holds for the class of  $L^1$ -spaces; see Lemma 4.2.

#### 4. APPLICATION: REPRESENTATIONS OF $C^*$ -ALGEBRAS ON $L^p$ -SPACES

The goal of this section is to show that a  $C^*$ -algebra is necessarily commutative if it has an isometric representation on an  $L^p$ -space for  $p \in [1, \infty) \setminus \{2\}$ ; see Theorem 4.4. This result is a crucial ingredient to obtain the results in [GT16c], where it is shown that algebras of convolution operators on  $L^p(G)$ , for a nontrivial locally compact group  $G$ , are not presentable on an  $L^q$ -space unless  $p = q$ , or  $p$  and  $q$  are conjugate Hölder exponents, or  $p = 2$  and  $G$  is commutative.

To main ingredients to prove Theorem 4.4 are: First, we reduce the problem to the unital case by showing that a  $C^*$ -algebra has an isometric representation on an  $L^p$ -space if and only if its multiplier algebra has a unital, isometric representation on an  $L^p$ -space; see Lemma 4.2. Second, given a  $\sigma$ -unital measure space  $\mu$ , we use

Lamperti's theorem to show that every hermitian operator on  $L^p(\mu)$  is a multiplication operator, which implies that all hermitian operators on  $L^p(\mu)$  commute; see Lemma 4.3.

**Lemma 4.1.** *Let  $p \in [1, \infty)$ , and let  $X$  be an  $L^p$ -space. Then  $X$  does not contain an isomorphic copy of  $c_0$ .*

*Proof.* This follows easily from the following facts: Every  $L^p$ -space is weakly sequentially complete; if a space  $Y$  is weakly sequentially complete, then so is every closed subspace of  $Y$  and every space isomorphic to  $Y$ ; moreover,  $c_0$  is not weakly sequentially complete.

For  $p > 1$ , the statement also follows using that  $L^p$ -spaces are reflexive, and reflexivity is an isomorphism invariant that passes to closed subspaces, and  $c_0$  is not reflexive.  $\square$

For  $p > 1$ , the following result also follows from Theorem 3.19.

**Lemma 4.2.** *Let  $p \in [1, \infty)$ , and let  $A$  be a  $C^*$ -algebra with an isometric representation on an  $L^p$ -space (that is,  $A$  is an  $L^p$ -operator algebra). Then there is a nondegenerate isometric representation of  $A$  on an  $L^p$ -space. It follows that the multiplier algebra of  $A$  has a unital, isometric representation on an  $L^p$ -space.*

*Proof.* Choose an  $L^p$ -space  $X$  and an isometric representation  $\varphi: A \rightarrow \mathcal{B}(X)$ . By Lemma 4.1,  $X$  does not contain an isomorphic copy of  $c_0$ . It follows from Proposition 3.15, that the essential space  $X_\varphi$  of  $\varphi$  is 1-complemented in  $X$ . We may then argue as in the proof of Theorem 3.19 to deduce that  $X_\varphi$  is an  $L^p$ -space and that the restriction of  $\varphi$  to  $X_\varphi$  extends to a unital isometric representation of the (left) multiplier algebra of  $A$  on  $X_\varphi$ .  $\square$

Given a unital Banach algebra  $A$ , recall that  $a \in A$  is said to be *hermitian* if  $\|\exp(ita)\| = 1$  for every  $t \in \mathbb{R}$ . (Equivalently,  $\|\exp(ita)\| \leq 1$  for every  $t \in \mathbb{R}$ .) An element of a unital  $C^*$ -algebra is hermitian if and only if it is self-adjoint.

Let  $A$  and  $B$  be unital Banach algebras, let  $\varphi: A \rightarrow B$  be a multiplicative operator, and let  $a \in A$  be hermitian. If  $\varphi$  is unital and contractive, then  $\varphi(a)$  is hermitian in  $B$ . Indeed, using that  $\varphi$  is unital, we have  $\exp(it\varphi(a)) = \varphi(\exp(ita))$  for every  $t$ . Further, using that  $\varphi$  is contractive, we obtain that

$$\|\exp(it\varphi(a))\| = \|\varphi(\exp(ita))\| \leq \|\exp(ita)\| = 1,$$

as desired. If  $\varphi$  is not unital or not contractive, then  $\varphi(a)$  need not be hermitian.

The following result is probably known, but we could not locate a reference. For the case  $\ell_p$ , it has appeared in [Tam69, Theorem 2], see also [BS74, Remarks p.35].

**Lemma 4.3.** *Let  $p \in [1, \infty) \setminus \{2\}$ , let  $\mu$  be a complete,  $\sigma$ -finite measure space, and let  $a \in \mathcal{B}(L^p(\mu))$ . Then  $a$  is hermitian if and only if  $a$  is the multiplication operator corresponding to some bounded, measurable function  $X \rightarrow \mathbb{R}$ . In particular, all hermitian operators on  $L^p(\mu)$  commute.*

*Proof.* Assume that  $a$  is given by multiplication with a bounded, measurable function  $h: X \rightarrow \mathbb{R}$ . We need to show that  $\|\exp(ita)\| \leq 1$  for every  $t \in \mathbb{R}$ . Let  $t \in \mathbb{R}$ . Then the operator  $\exp(ita)$  is given by multiplication with the function  $\exp(ith)$ . For each  $x \in X$ , we have  $\exp(ith)(x) \in S^1$ , and therefore  $\|\exp(ith)\|_\infty \leq 1$ . It follows that  $\|\exp(ita)\| \leq 1$ , as desired.

To show the converse implication, assume that  $a$  is hermitian. By rescaling, if necessary, we may assume that  $\|a\| \leq \frac{\pi}{2}$ . Let  $\text{Isom}(L^p(\mu))$  denote the set of surjective isometric operators  $L^p(\mu) \rightarrow L^p(\mu)$ . For  $t \in \mathbb{R}$ , set  $u_t := \exp(ita)$ . Since  $a$  is hermitian, we have  $\|u_t\| \leq 1$  for every  $t$ . Moreover, we have  $u_t u_{-t} = u_{-t} u_t = \text{id}$ ,

which implies that  $u_t$  belongs to  $\text{Isom}(L^p(\mu))$  for every  $t$ . Moreover, the map  $[0, 1] \rightarrow \text{Isom}(L^p(\mu))$ ,  $t \mapsto u_t$ , is a norm-continuous path.

By Lamperti's theorem (in the form given in Theorem 4.4 in [GT15a]; see also [Lam58] for the original statement), for every  $v \in \text{Isom}(L^p(\mu))$  there exists a measurable function  $h_v: X \rightarrow S^1$  and a measure class preserving automorphism  $T_v: X \rightarrow X$  (see Definition 4.1 in [GT15a]) such that

$$(v\xi)(x) = h_v(x) \left( \frac{d(\mu \circ T_v^{-1})}{d\mu}(x) \right)^{\frac{1}{p}} \xi(T_v^{-1}(x))$$

for all  $\xi \in L^p(\mu)$  and  $\mu$ -almost every  $x \in X$ . (The transformation  $T_v$  is called the *spatial realization* of  $v$  in [Phi12].)

We may apply Lamperti's theorem to  $u_t$ , for each  $t \in [0, 1]$ . By Lemma 6.22 in [Phi12], if  $[0, 1] \rightarrow \text{Isom}(L^p(\mu))$ ,  $t \mapsto u_t$ , is a norm-continuous map, then the spatial realization of  $u_0$  and  $u_1$  agree. Since  $u_0$  is the identity map on  $L^p(\mu)$ , its spatial realization is the identity map on  $X$ . Hence the spatial realization of  $u_1$  is the identity map as well. Thus, there exists a measurable function  $h: X \rightarrow S^1$  such that  $u_1$  is given by multiplication with  $h$ .

Set  $T := \{it : t \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$ , and set  $P := \{z \in S^1 : \text{Re}(z) \geq 0\}$ . The restriction of the exponential map to  $T$  is a bijection onto  $P$ . We let  $\log: P \rightarrow T$  denote the inverse map, which is contained in a branch of the logarithm and is therefore analytic on a neighborhood of  $P$ .

Let  $\sigma(a)$  denote the spectrum of  $a$ . Since  $\|a\| \leq \frac{\pi}{2}$ , the spectrum of  $ia$  is contained in  $T$ , and consequently  $\sigma(u_1) \subseteq P$ . Thus, we may apply (analytic) functional calculus to  $u_1$  and obtain that  $ia = \log(u_1)$ . Recall that  $u_1$  is given by multiplication with the function  $h: X \rightarrow S^1$ . After changing  $h$  on a null-set, we may assume that  $h$  takes image in  $P$ . It follows that  $ia$  is given by multiplication by the function  $\log(h)$ , which takes values in  $T$ . This implies that  $a$  is given by multiplication by the function  $-i \log(h)$ , which is a bounded, measurable real-valued function, as desired.  $\square$

**Theorem 4.4.** *Let  $A$  be a  $C^*$ -algebra and let  $p \in [1, \infty) \setminus \{2\}$ . Then  $A$  can be isometrically represented on an  $L^p$ -space if and only if  $A$  is commutative.*

*Proof.* Let  $A$  be a commutative  $C^*$ -algebra. We may assume that  $A$  is unital. By the Gelfand representation theorem, there exists a compact Hausdorff space  $X$  such that  $A \cong C(X)$ . Given a positive Borel measure  $\mu$  on  $X$ , we consider the representation  $\varphi_\mu$  of  $C(X)$  as multiplication operators on  $L^p(\mu)$ . In general, it may not be possible to choose a single measure  $\mu$  such that  $\varphi_\mu$  is isometric. (This is the same for  $p = 2$  and corresponds to the question whether  $C(X)$  admits a faithful state.) However, just as for representations on Hilbert spaces, it is always possible to choose a family  $(\mu_j)_j$  of measures and consider the diagonal representation  $\varphi := \prod_j \varphi_{\mu_j}$  on the  $L^p$ -space  $p - \oplus_j L^p(\mu_j)$  such that  $\varphi$  is isometric.

To show the converse implication, assume that  $A$  has an isometric representation on an  $L^p$ -space. By Lemma 4.2, there is a unital, isometric representation of the multiplier algebra  $M(A)$  on an  $L^p$ -space. Since  $A$  is commutative if (and only if)  $M(A)$  is commutative, we can assume that  $A$  and its representation are unital. Since any element in a  $C^*$ -algebra is a linear combination of two self-adjoint elements, it is enough to show that self-adjoint elements in  $A$  commute with each other.

Let  $a, b \in A_{\text{sa}}$ . We claim that  $ab = ba$ . Since  $C^*(1, a, b)$ , the unital sub- $C^*$ -algebra generated by  $a$  and  $b$ , is a separable unital  $C^*$ -algebra, we may assume that  $A$  itself is separable. In this case, by Proposition 1.25 in [Phi13], there exist a separable  $L^p$ -space  $E$  and a nondegenerate (and hence automatically unital) isometric representation  $\varphi: A \rightarrow \mathcal{B}(E)$ . Every separable  $L^p$ -space is isometrically isomorphic

to  $L^p(\mu)$  for some complete,  $\sigma$ -finite measure space  $\mu$ . The image of a hermitian element under a unital, contractive homomorphism is again hermitian. Thus,  $\varphi(a)$  and  $\varphi(b)$  are hermitian. It follows from Lemma 4.3 that  $\varphi(a)$  and  $\varphi(b)$  commute. Since  $\varphi$  is isometric, we deduce that  $a$  and  $b$  commute, as desired.  $\square$

**Remarks 4.5.** (1) Theorem 4.4 shows that noncommutative  $C^*$ -algebras have no isometric representations on  $L^p$ -spaces, for  $p \in [1, \infty) \setminus \{2\}$ . This is very different for noncommutative  $L^p$ -spaces. (We refer to [PX03] for definitions and basic facts of noncommutative  $L^p$ -spaces.)

Given a Hilbert space  $H$ , and  $p \in [1, \infty)$ , consider the associated noncommutative  $L^p$ -space  $\mathcal{S}_p(H)$  of Schatten- $p$  class operators. Using that  $\mathcal{S}_p(H)$  is a (nonclosed) ideal in  $\mathcal{B}(H)$ , we may define  $\varphi: \mathcal{B}(H) \rightarrow \mathcal{B}(\mathcal{S}_p(H))$  by  $\varphi(a)b := ab$ , for  $a \in \mathcal{B}(H)$  and  $b \in \mathcal{S}_p(H)$ . It is well-known that  $\varphi$  is an isometric representation. Since every  $C^*$ -algebra embeds as a closed subalgebra of  $\mathcal{B}(H)$ , for some Hilbert space  $H$ , it follows that every  $C^*$ -algebra has an isometric representation on a noncommutative  $L^p$ -spaces, for every  $p \in [1, \infty)$ .

(2) One important ingredient in the proof of Theorem 4.4 was that all hermitian operators on  $L^p(\mu)$  commute, for a complete,  $\sigma$ -finite measure space  $\mu$  and  $p \in [1, \infty) \setminus \{2\}$ . We remark that there are other Banach spaces  $X$  such that all hermitian operators on  $X$  commute. There are even Banach spaces  $X$  such that every hermitian operator on  $X$  is a multiple of the identity operator; see [BS74, Section 3]. For such  $X$ , there is no isometric, nondegenerate representation of any nonzero  $C^*$ -algebra  $A \neq \mathbb{C}$  on  $X$ .

## 5. APPLICATION: UNIVERSAL COMPLETIONS FOR REPRESENTATIONS ON $L^p$ -SPACES

**5.1.** Let  $A$  be a Banach algebra, and let  $\mathcal{E}$  be a class of Banach spaces. We let  $\text{Rep}_{\mathcal{E}}(A)$  denote the class of all non-degenerate, contractive representations  $A \rightarrow \mathcal{B}(E)$ , for  $E$  ranging over Banach spaces in  $\mathcal{E}$ . Given  $a \in A$ , we set

$$\|a\|_{\mathcal{E}} := \sup \{ \|\varphi(a)\| : \varphi \in \text{Rep}_{\mathcal{E}}(A) \}.$$

We let  $F_{\mathcal{E}}(A)$  denote the completion of  $A$  with respect to the norm  $\|\cdot\|_{\mathcal{E}}$  (with the convention that we first quotient out the ideal consisting of those  $a \in A$  with  $\|a\|_{\mathcal{E}} = 0$ .) The multiplication on  $A$  extends to  $F_{\mathcal{E}}(A)$  and turns it into a Banach algebra. Note that  $F_{\mathcal{E}}(A)$  captures the representation theory of  $A$  on Banach spaces in  $\mathcal{E}$ .

**Theorem 5.2.** *Let  $\mathcal{E}$  be a class of reflexive Banach spaces that is closed under passing to 1-complemented subspaces (for instance, the class of  $L^p$ -spaces, for some fixed  $p \in (1, \infty)$ ), and let  $A$  be a Banach algebra with a contractive left approximate identity. Then*

$$\|a\|_{\mathcal{E}} = \sup \{ \|\varphi(a)\| : \varphi: A \rightarrow \mathcal{B}(E) \text{ contractive representation, } E \in \mathcal{E} \},$$

*for every  $a \in A$ . That is, the universal completion of  $A$  with respect to contractive representations on spaces in  $\mathcal{E}$  is the same as the universal completion of  $A$  with respect to nondegenerate contractive representations on spaces in  $\mathcal{E}$ .*

*Proof.* Let  $a \in A$ . Let  $K$  denote the supremum of  $\|\varphi(a)\|$  for contractive representations  $\varphi: A \rightarrow \mathcal{B}(E)$  with  $E \in \mathcal{E}$ . We clearly have  $\|a\|_{\mathcal{E}} \leq K$ .

To show the converse inequality, let  $X \in \mathcal{E}$  and let  $\varphi: A \rightarrow \mathcal{B}(X)$  be a contractive representations. We need to verify that  $\|\varphi(a)\| \leq \|a\|_{\mathcal{E}}$ . Let  $X_{\varphi}$  be the essential space of  $\varphi$ . It follows from Corollary 3.13 that  $X_{\varphi}$  is 1-complemented in  $X$ . By assumption,  $X_{\varphi}$  belongs to  $\mathcal{E}$ . Let  $\varphi_0$  denote the restriction of  $\varphi$  to  $X_{\varphi}$ . Note

that  $\|\varphi(b)\| = \|\varphi_0(b)\|$  for every  $b \in A$ . Thus,  $\varphi_0$  is a nondegenerate, contractive representation of  $A$  on a space in  $\mathcal{E}$ . Using this at the second step, we deduce

$$\|\varphi(a)\| = \|\varphi_0(a)\| \leq \|a\|_{\mathcal{E}},$$

as desired.  $\square$

Let  $G$  be a locally compact group. We equip  $G$  with a fixed left invariant Haar measure, and we let  $L^1(G)$  denote the corresponding group algebra of integrable functions, with product given by convolution. Note that  $L^1(G)$  is a Banach algebra with a contractive, approximate identity.

Let  $E$  be a Banach space, and let  $\text{Isom}(E)$  denote the group of surjective isometric operators  $E \rightarrow E$ . We equip  $\text{Isom}(E)$  with the strong operator topology. An *isometric representation* of  $G$  on  $E$  is a continuous group homomorphism  $G \rightarrow \text{Isom}(E)$ . Every isometric representation of  $G$  on  $E$  can be integrated to obtain a nondegenerate contractive representation  $L^1(G) \rightarrow \mathcal{B}(E)$ . It is well-known that this induces a bijection between isometric representations of  $G$  on  $E$  and nondegenerate, contractive representations  $L^1(G) \rightarrow \mathcal{B}(E)$ ; see for example [GT15b, Proposition 2.4].

Given a class  $\mathcal{E}$  of Banach spaces, we denote  $F_{\mathcal{E}}(L^1(G))$  by  $F_{\mathcal{E}}(G)$ . The Banach algebra  $F_{\mathcal{E}}(G)$  captures the representation theory of  $G$  on Banach spaces in  $\mathcal{E}$ . Given  $p \in [1, \infty)$ , let  $L^p$  denote the class of  $L^p$ -spaces. We set  $F^p(G) := F_{L^p}(G)$ , and we call  $F^p(G)$  the *universal group  $L^p$ -operator algebra*. Note that  $L^2$  is the class of Hilbert spaces and that  $F^2(G)$  is the universal group  $C^*$ -algebra of  $G$ , usually denoted by  $C^*(G)$ . The Banach algebras  $F^p(G)$  have been studied in [GT15b].

**Corollary 5.3.** *Let  $p \in (1, \infty)$ , and let  $G$  be a locally compact group. Then for each  $f \in L^1(G)$ , the norm  $\|f\|_{L^p}$  agrees with the supremum of  $\|\varphi(f)\|$  for all (not necessarily nondegenerate) contractive representations  $\varphi: L^1(G) \rightarrow \mathcal{B}(E)$  on some  $L^p$ -space  $E$ .*

*That is, the universal group  $L^p$ -operator algebra  $F^p(G)$  is also universal for (possibly degenerate) contractive representations of  $L^1(G)$  on  $L^p$ -spaces.*

*Analogous statements hold for the universal group  $SL^p$ -operator algebra (group  $QL^p$ -operator algebra, group  $QSL^p$ -operator algebra).*

**Remark 5.4.** Corollary 5.3 is a crucial ingredient in [GT16c], where it is used to reduce the question of representability of certain Banach algebras of convolution on  $L^q$ -spaces to a question about *nondegenerate* representations. In particular, for every nontrivial, locally compact group  $G$ , it follows that the reduced group  $L^p$ -operator algebras  $F_{\lambda}^p(G)$  and  $F_{\lambda}^q(G)$  are not isometrically isomorphic as Banach algebras for  $p, q \in [1, 2]$  with  $p \neq q$ .

## REFERENCES

- [ADG72] C. A. AKEMANN, P. G. DODDS, and J. L. B. GAMLEN, Weak compactness in the dual space of a  $C^*$ -algebra, *J. Functional Analysis* **10** (1972), 446–450. MR 0344898.
- [BS74] E. BERKSON and A. SOUROUT, The Hermitian operators on some Banach spaces, *Studia Math.* **52** (1974), 33–41. MR 0355668 (50 #8142). Zbl 0258.47026.
- [BPc58] C. BESSAGA and A. PEŁ CZYŃSKI, On bases and unconditional convergence of series in Banach spaces, *Studia Math.* **17** (1958), 151–164. MR 0115069. Zbl 0084.09805.
- [Bla06] B. BLACKADAR, *Operator algebras*, *Encyclopaedia of Mathematical Sciences* **122**, Springer-Verlag, Berlin, 2006, Theory of  $C^*$ -algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. MR 2188261 (2006k:46082). Zbl 1092.46003.
- [Con90] J. B. CONWAY, *A course in functional analysis*, second ed., *Graduate Texts in Mathematics* **96**, Springer-Verlag, New York, 1990. MR 1070713. Zbl 0706.46003.
- [Daw10] M. DAWS, Multipliers, self-induced and dual Banach algebras, *Dissertationes Math. (Rozprawy Mat.)* **470** (2010), 62. MR 2681109 (2011i:46048). Zbl 1214.43004.

- [GT15a] E. GARDELLA and H. THIEL, Banach algebras generated by an invertible isometry of an  $L^p$ -space, *J. Funct. Anal.* **269** (2015), 1796–1839. MR 3373434. Zbl 1334.46040.
- [GT15b] E. GARDELLA and H. THIEL, Group algebras acting on  $L^p$ -spaces, *J. Fourier Anal. Appl.* **21** (2015), 1310–1343. MR 3421918. Zbl 1334.22007.
- [GT16a] E. GARDELLA and H. THIEL, Preduals and complementation of spaces of bounded linear operators, preprint (arXiv:1609.05326 [math.FA]), 2016.
- [GT16b] E. GARDELLA and H. THIEL, Quotients of Banach algebras acting on  $L^p$ -spaces, *Adv. Math.* **296** (2016), 85–92. MR 3490763. Zbl 1341.47089.
- [GT16c] E. GARDELLA and H. THIEL, Representations of  $p$ -convolution algebras on  $l^q$ -spaces, preprint (arXiv:1609.08612 [math.FA]), 2016.
- [GK16] S. GHASEMI and P. KOSZMIDER, An extension of compact operators by compact operators with no nontrivial multipliers, preprint (arXiv:1609.04766 [math.OA]), 2016.
- [Joh64] B. E. JOHNSON, An introduction to the theory of centralizers, *Proc. London Math. Soc.* (3) **14** (1964), 299–320. MR 0159233. Zbl 0143.36102.
- [Lam58] J. LAMPERTI, On the isometries of certain function-spaces, *Pacific J. Math.* **8** (1958), 459–466. MR 0105017. Zbl 0085.09702.
- [LT71] J. LINDENSTRAUSS and L. TZAFRIRI, On the complemented subspaces problem, *Israel J. Math.* **9** (1971), 263–269. MR 0276734 (43 #2474). Zbl 0211.16301.
- [NR11] M. NEAL and B. RUSSO, Existence of contractive projections on preduals of JBW\*-triples, *Israel J. Math.* **182** (2011), 293–331. MR 2783974. Zbl 1232.46063.
- [Pal94] T. W. PALMER, *Banach algebras and the general theory of \*-algebras. Vol. I, Encyclopedia of Mathematics and its Applications* **49**, Cambridge University Press, Cambridge, 1994, Algebras and Banach algebras. MR 1270014 (95c:46002).
- [Pfi94] H. PFITZNER, Weak compactness in the dual of a  $C^*$ -algebra is determined commutatively, *Math. Ann.* **298** (1994), 349–371. MR 1256621. Zbl 0791.46035.
- [Phi12] N. C. PHILLIPS, Analogs of cuntz algebras on  $L^p$  spaces, preprint (arXiv:1201.4196 [math.FA]), 2012.
- [Phi13] N. C. PHILLIPS, Crossed products of  $L^p$  operator algebras and the  $K$ -theory of cuntz algebras on  $L^p$  spaces, preprint (arXiv:1309.6406 [math.FA]), 2013.
- [PX03] G. PISIER and Q. XU, Non-commutative  $L^p$ -spaces, in *Handbook of the geometry of Banach spaces, Vol. 2*, North-Holland, Amsterdam, 2003, pp. 1459–1517. MR 1999201. Zbl 1046.46048.
- [Spa76] P. G. SPAIN, A generalisation of a theorem of Grothendieck, *Quart. J. Math. Oxford Ser. (2)* **27** (1976), 475–479. MR 0442626. Zbl 0341.46007.
- [Spa15] P. G. SPAIN, Representations of  $C^*$ -algebras in dual & right dual Banach algebras, *Houston J. Math.* **41** (2015), 231–263. MR 3347946. Zbl 06522521.
- [Tam69] K. W. TAM, Isometries of certain function spaces, *Pacific J. Math.* **31** (1969), 233–246. MR 0415295. Zbl 0189.43104.
- [Tay72] D. C. TAYLOR, A general Phillips theorem for  $C^*$ -algebras and some applications, *Pacific J. Math.* **40** (1972), 477–488. MR 0308799 (46 #7913). Zbl 0239.46061.
- [Tza69] L. TZAFRIRI, Remarks on contractive projections in  $L_p$ -spaces, *Israel J. Math.* **7** (1969), 9–15. MR 0248514 (40 #1766). Zbl 0184.15103.
- [Whi66] R. WHITLEY, Mathematical Notes: Projecting  $m$  onto  $c_0$ , *Amer. Math. Monthly* **73** (1966), 285–286. MR 1533692. Zbl 0143.15301.

EUSEBIO GARDELLA MATHEMATISCHES INSTITUT, FACHBEREICH MATHEMATIK UND INFORMATIK DER UNIVERSITÄT MÜNSTER, EINSTEINSTRASSE 62, 48149 MÜNSTER, GERMANY.

*E-mail address:* [gardella@uni-muenster.de](mailto:gardella@uni-muenster.de)

*URL:* <http://pages.uoregon.edu/gardella/>

HANNES THIEL MATHEMATISCHES INSTITUT, FACHBEREICH MATHEMATIK UND INFORMATIK DER UNIVERSITÄT MÜNSTER, EINSTEINSTRASSE 62, 48149 MÜNSTER, GERMANY.

*E-mail address:* [hannes.thiel@uni-muenster.de](mailto:hannes.thiel@uni-muenster.de)

*URL:* [www.math.uni-muenster.de/u/hannes.thiel/](http://www.math.uni-muenster.de/u/hannes.thiel/)